

ON ZARISKI'S MULTIPLICITY CONJECTURE

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1. INTRODUCTION

Since the early 1960's, O. Zariski developed a comprehensive theory of equisingularity in codimension one. He initiated an equisingularity program with topological, differential geometrical and purely algebraical point of view and proposed a problem list in [22] as an extraction of many possible conjectures in singularity theory [23]. In this part we will be concerned with topological aspects of this program and more specifically with the so-called Zariski's multiplicity conjecture. We first recall some definitions. Let $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions and V_f and V_g be two germs at the origin of the hypersurfaces defined by $f^{-1}(0)$ and $g^{-1}(0)$ respectively. We suppose $0 \in \mathbb{C}^n$ is an isolated singularity of the functions. The *algebraic multiplicity* m_f of the germs of V_f or f is the order of vanishing of function f at $0 \in \mathbb{C}^n$ or equivalently is the order of the first nonzero leading term in the Taylor expansion of f

$$f = f_\nu + f_{\nu+1} + \cdots$$

where f_i is homogeneous polynomial of degree i .

Definition 1. We say V_f and V_g are *topologically equisingular* or topologically V -equivalent if there is a germ of homeomorphism $\phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ sending V_f onto V_g . More precisely, there are neighborhoods U and U' of $0 \in \mathbb{C}^n$ such that f and g are defined and a homeomorphism $\phi : U \rightarrow U'$ such that $\phi(f^{-1}(0) \cap U) = g^{-1}(0) \cap U'$ and $\phi(0) = 0$.

Zariski conjecture. *Topological equisingularity of germs of hypersurfaces implies equimultiplicity.*

A well known result by Burau [4] and Zariski [23] states an affirmative answer in the case of curves ($n = 2$). In higher dimension the conjecture is still open despite more than three decades effort to prove it.

Here we discuss some features of the problem, especially the relations of the work of A'Campo on the zeta function of a monodromy and the Zariski's multiplicity conjecture. Also some previous results are sharpened;

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the results of [6] and [18] in theorem (3.2) and the one in [7] in theorem (5.2). In an analogy with hypersurfaces, J.F. Mattei asked the same question about multiplicity for holomorphic foliations. In section (6) we recall some remarkable results for foliations.

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2. PRELIMINARIES

In [15], Milnor has opened a beautiful account on the complex hypersurfaces. The main achievement of it, is the Milnor fibration which we mention here. Also we recall briefly some generalities about complex hypersurfaces.

Let $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ be a holomorphic function on an open neighborhood of 0 in \mathbb{C}^n and $f(0) = 0$. We denote $D_\epsilon = \{z \in \mathbb{C}^n : \|z\| \leq \epsilon\}$, $S_\epsilon = \partial D_\epsilon$, $H_0 = \{z \in \mathbb{C}^n | f(z) = 0\}$ and $d_z f = (\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z))$.

We say the origin is an isolated singularity of f if $d_0 f = 0$ and $d_z f \neq 0$ for a neighborhood of $0 \in \mathbb{C}^n$ except 0.

Let \mathcal{O}_n be the ring of germs of holomorphic functions defined in some neighborhood of $0 \in \mathbb{C}^n$ and let $\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \rangle$ be the ideal generated by the germs at $0 \in \mathbb{C}^n$ of derivative components of f . We define *Milnor number* μ of the holomorphic function f at $0 \in \mathbb{C}^n$ as

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_n / \langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \rangle$$

This number is finite and nonzero if and only if $0 \in \mathbb{C}^n$ is an isolated singularity of f , a hypothesis which we will assume from now on. In this case μ coincides with the topological degree of the Gauss mapping induced by $d_z f$ on S_ϵ for ϵ small enough. The following lemma is useful to deal with the Milnor number.

Lemma 2.1. *Let $0 < \mu < \infty$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for any $c \in \mathbb{C}^n$ with $\|c\| < \delta$ the number of solutions of the equation $d_z f = c$ in the ball D_ϵ is at most μ . Moreover, if p_1, \dots, p_m , $m \leq \mu$, are such solutions, then $\sum_{i=1}^m \mu(f - \sum_{i=1}^n z_i c_i, p_i) = \mu$.*

The following theorem is called the Milnor fibration theorem:

Theorem 2.2. *For ϵ small enough the mapping $\psi_\epsilon : S_\epsilon \setminus H_0 \rightarrow S^1$ defined by $\psi_\epsilon(z) = f(z)/\|f(z)\|$ is a smooth fibration which is called the Milnor fibration. Moreover the fibers of ψ_ϵ have the homotopy type of a bouquet of μ (the Milnor number of the holomorphic function f at $0 \in \mathbb{C}^n$) spheres of dimension $n - 1$.*

Also we call the number of spheres, the number of *vanishing cycles* of f at 0. The following theorem is due to Lê [14]:

Theorem 2.3. *If V_f and V_g are topologically equisingular then the number of vanishing cycles at 0 of f and g are the same.*

Now we recall some definitions and facts about deformations of functions. A *deformation* of a holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is a family $(f_t)_{t \in [0,1]}$ of germs of holomorphic functions with isolated singularities at $0 \in \mathbb{C}^n$. The *jump of the family* (f_t) is $\mu(f_0) - \mu(f_t)$, where μ is the Milnor number at the origin. It is independent of t for t small enough, moreover by the upper semi-continuity of μ this number is a non-negative integer.

We use frequently the following theorem proved by Lê and Ramanujam [12]:

Theorem 2.4. *Let $(f_s)_{s \in [0,1]}$ be a C^∞ family of hypersurfaces having an isolated singularity at the origin. If the Milnor number of singularity does not change then the topological type of singularity does not change too provided that $n \neq 3$.*

In [11], theorem (2.4) is generalized which we recall in section (5). Finally we recall an interesting result of P. Samuel [19]:

Theorem 2.5. *Every germ V_f is analytically equivalent with V_g in which g is a polynomial.*

Moreover we may choose a polynomial g with cutting the Taylor expansion of f in somewhere. By the theorem of (2.5) it is enough to consider polynomials to prove the conjecture.

3. THE TOPOLOGICAL RIGHT EQUIVALENT COMPLEX HYPERSURFACES

In this section we recall several ways to define a topological type of a holomorphic function and relations between them according to [10], [17] and [20].

Let $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be two germs of holomorphic functions with an isolated singularity at the origin.

Definition 2. f and g are topologically right equivalent if there is a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ satisfying $f = g \circ \varphi$

Definition 3. f and g are topologically right-left equivalent if there are germs of homeomorphism $\varphi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ satisfying $f = \psi \circ g \circ \varphi$

Put $V_f := f^{-1}(0)$. By [15], $S_\varepsilon^{2n-1} \cap V_f$ is a smooth $(2n-3)$ -dimensional manifold for $\varepsilon > 0$ sufficiently small. The pair $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is called the link of the singularity of f .

Definition 4. f and g are link equivalent if $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_g)$ for all sufficiently small ε .

By the definitions, the right equivalence implies the right-left equivalence, which in turn implies the V-equivalence. The outstanding result, obtained by King [10] in $n \neq 3$ and by Perron [17] in $n = 3$, is the following:

Theorem 3.1. *The topological V-equivalence implies topologically right-left equivalence. Moreover if f and g are topologically right-left equivalent then g is topologically right equivalent either to f or to \bar{f} , the complex conjugate of f .*

Using theorem (3.1), Risler and Trotman in [18] proved that right-left bilipschitz equivalence implies equimultiplicity.

Since $(D_\varepsilon^{2n-1}, D_\varepsilon^{2n-1} \cap V_f)$ is homeomorphic to the cone cover over the link $(S_\varepsilon^{2n-1}, S_\varepsilon^{2n-1} \cap V_f)$ ([15]), the link equivalence implies the V-equivalence. Conversely Saeki [20] showed that the topological V-equivalence implies the link equivalence. This means that there is a homeomorphism $\varphi_1 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$, sending V_f onto V_g and such that $|\varphi_1(z)| = |z|$. By theorem (3.1) there is a homeomorphism $\varphi_2 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $|f(z)| = |g \circ \varphi_2(z)|$. Comte, Milman and Trotman [6] showed that if there is a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ having *simultaneously* the properties of φ_1 and of φ_2 then the multiplicity conjecture is true. In fact they proved that it suffices to assume that there are positive constants A, B, C and D such that:

- (1) $A|z| \leq |\varphi(z)| \leq B|z|$, for all z near 0, and
- (2) $C|f(z)| \leq |g \circ \varphi(z)| \leq D|f(z)|$, for all z near 0.

Now we prove that it is enough to assume the conditions (1) and (2) are valid for some special sequences converge to the origin. Given two holomorphic function germs $f, g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, by an analytic change of coordinates one may assume that the z_1 -axis is not contained in the tangent cones $C(V_f), C(V_g)$ (respectively the zero set of first non zero jet of f and g), so that $f(z_1, 0, \dots, 0) \neq 0$ and $g(z_1, 0, \dots, 0) \neq 0$ for a neighborhood of 0 in the z_1 -axis, and by theorem (2.5) one may assume f and g are polynomials. In this situation we have the following:

Theorem 3.2. *Suppose there are a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ with inverse ψ and positive constants A, B, C and D and two sequences w_m and w'_m in the z_1 -axis which converge to the origin with the following properties:*

- (i) $|\psi(w'_m)| \leq A|w'_m|$, $|\varphi(w_m)| \leq B|w_m|$ and
 - (ii) $C|f(w_m)| \leq |g \circ \varphi(w_m)|$, $D|g(w'_m)| \leq |f \circ \psi(w'_m)|$
- then $m_f = m_g$.

The conditions (i) and (ii) are slightly weaker than conditions (1) and (2) above.

Proof. Write

$$\begin{aligned} f(z) &= f_k(z) + f_{k+1}(z) + \dots + f_{k+r}(z), \\ g(z) &= g_l(z) + g_{l+1}(z) + \dots + g_{l+s}(z). \end{aligned}$$

f_i and g_j are homogeneous parts of degree i and j of f and g respectively. f_k and g_l are not identically zero. We want to prove $k = l$. By contrary suppose $l > k$. The other case is similar. Let $w_1 = (z_1, 0, \dots, 0)$ and

$w_m = (t_m z_1, 0, \dots, 0)$, $t_m \neq 0$ and converges to the origin. Also write g in the following form:

$$g(z) = \sum_{j=l}^{l+s} \sum_{|\beta|=j} C_\beta^j z^\beta,$$

where $z = (z_1, \dots, z_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \in \mathbb{N} \cup \{0\}$. Now we have

$$f(w_m) = f_k(w_m) + f_{k+1}(w_m) + \dots + f_{k+r}(w_m) \text{ or}$$

$$f(w_m) = t_m^k [f_k(w_1) + t_m f_{k+1}(w_1) + \dots + t_m^r f_{k+r}(w_1)]$$

and

$$|(g \circ \varphi)(w_m)| \leq \sum_{j=l}^{l+s} \sum_{|\beta|=j} |C_\beta^j| B^j |t_m z_1|^j$$

by (i). Now we use condition (ii). It is:

$$C|f(w_m)| \leq |g \circ \varphi(w_m)| \text{ or}$$

$$C|t_m^k [f_k(w_1) + t_m f_{k+1}(w_1) + \dots + t_m^r f_{k+r}(w_1)]| \leq \sum_{j=l}^{l+s} \sum_{|\beta|=j} |C_\beta^j| B^j |t_m z_1|^j.$$

Divided two sides of above inequality by $|t_m^k|$ we obtain the following:

$$C|f_k(w_1) + t_m f_{k+1}(w_1) + \dots + t_m^r f_{k+r}(w_1)| \leq \sum_{j=l}^{l+s} \sum_{|\beta|=j} |C_\beta^j| B^j |t_m z_1|^{j-k},$$

or

$$C|f_k(w_1)| \leq \sum_{j=l}^{l+s} \sum_{|\beta|=j} |C_\beta^j| B^j |t_m z_1|^{j-k} + C|t_m^1 f_{k+1}(w_1) + \dots + t_m^r f_{k+r}(w_1)|.$$

When t_m goes to zero, the right hand of the last inequality goes to zero but the left hand is a positive constant. This contradiction shows $l = k$. \square

4. THE ZETA FUNCTION OF A MONODROMY

Now we recall some features from [1] and [2]. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial so that $f(0) = 0$ and consider the hypersurface defined by it, $V_f = f^{-1}(0)$. The map

$$\pi : z \in S_\varepsilon^{2n-1} \setminus V_f \longmapsto \arg(f(z)) \in S^1,$$

defines a Milnor fibration of the hypersurface V_f at the origin. The fibre $F_\theta = \pi^{-1}(\theta)$, $\theta \in S^1$, is a $2(n-1)$ -dimensional differential manifold and the characteristic homeomorphism of this fibration

$$h : F_\theta \rightarrow F_\theta$$

is the geometric monodromy of V_f at the origin. By definition the zeta function of h is the following:

$$Z(t) = \prod_{q \geq 0} \{\det(Id^* - th^*; H^q(F_\theta, \mathbb{C}))\}^{(-1)^{q+1}}.$$

When the origin of \mathbb{C}^n is an isolated singular point of V_f , one has

$$H^q(F_\theta, \mathbb{C}) = \begin{cases} \mathbb{C} & q = 0 \\ 0 & q \neq 0, q \neq n \\ \mathbb{C}^\mu & q = n, \end{cases}$$

where μ is the Milnor number of f and therefore the characteristic polynomial $\Delta(t)$ of the monodromy at degree n is deduced from the zeta function $Z(t)$ by the formula

$$\Delta(t) = t^\mu \left[\frac{t-1}{t} Z\left(\frac{1}{t}\right) \right]^{(-1)^{n+1}}.$$

For an integer $k \geq 1$; let the integer number

$$\Lambda(h^k) = \sum_{q \geq 0} (-1)^q \text{Trace}[(h^*)^k; H^q(F_\theta, \mathbb{C})]$$

the Lefschetz number of the k -th power of h . Let s_1, s_2, \dots be the integers defined by the following recurrence relations:

$$\Lambda(h^k) = \sum_{i|k} s_i,$$

$k \geq 1$, then the zeta function of h is given by

$$Z(t) = \prod_{q \geq 0} (1 - t^i)^{\frac{-s_i}{i}}.$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of V_f , therefore the integers s_1, s_2, \dots are topological invariants.

Remark 4.1. In [2], A'Campo has calculated $\Lambda(h)$ as following:

$$\Lambda(h) = \begin{cases} 0 & \text{if } d_0 f = 0 \\ 1 & \text{if } d_0 f \neq 0. \end{cases}$$

This tells us that if f is regular and g is singular at the origin there is no topological equivalence between germs of V_f and V_g at the origin.

Remark 4.2. More generally Deligne has explained in a letter to A'Campo (see [1], [9]) that

$$\Lambda(h^k) = 0, \text{ if } 0 < k < \text{multiplicity of } V_f \text{ at the origin.}$$

The Lefschetz numbers $\Lambda(h^k)$ are topological invariants of the singularity of V_f , therefore the integers s_1, s_2, \dots are topological invariants. A'Campo discovered the meaning of the topological invariants s_1, s_2, \dots as following:

Let $\pi : X \rightarrow \mathbb{C}^n$ be a proper modification such that in all points of $S := \pi^{-1}(0)$, the divisor $V'_f := \pi^{-1}(V_f)$ has normal crossings. Such a local resolution of (\mathbb{C}^n, V_f) at the origin exists by the theorem of resolution of singularities due to Hironaka [8]. For every $m \in \mathbb{N}$, let S_m be all points $s \in S$ such that the equation of V'_f at s is of the form $z_1^m = 0$ for a local coordinate z of X at s and denote by $\chi(S_m)$ the Euler-Poincaré characteristic of S_m . A'Campo proved that $s_m = m\chi(S_m)$. More precisely:

Theorem 4.3. *One has*

- (1) $\Lambda(h^k) = \sum_{m|k} m\chi(S_m)$, $k \geq 1$,
- (2) $\Lambda(h^0) = \chi(F_\theta) = \sum_{m \geq 1} m\chi(S_m)$,
- (3) $\mu = \dim H^{n-1}(F_\theta, \mathbb{C}) = (-1)^{n-1}[-1 + \sum_{m \geq 1} m\chi(S_m)]$.

Therefore the numbers $\chi(S_m)$ don't depend on the chosen resolution and are topological invariants of the singularity. As a consequence we have the following result that may be useful for resolving the multiplicity conjecture.

Proposition 4.4. *If $f(z) = f_k(z) + f_{k+1}(z) + \dots + f_{k+r}(z)$, $g(z) = g_l(z) + g_{l+1}(z) + \dots + g_{l+s}(z)$ and $k+r < l$ then there is no topological equivalence between germs of V_f and V_g at the origin.*

Proof. Let h_1 and h_2 be the monodromies associated to f and g and s_1, s_2, \dots and s'_1, s'_2, \dots the two related sequences of f and g respectively as above. If there exists such an equivalence then $\Lambda(h_1^j) = \Lambda(h_2^j) = 0$ and $s_j = s'_j = 0$ for every j . Hence $\mu_f = \mu_g = (-1)^{n-1}[-1 + \sum_{j \geq 1} s_j] = (-1)^n$. If n is odd this is impossible and if n is even, then $\mu_f = \mu_g = 1$. In this case $k = l = 2$. Contradiction! \square

The second result is the following [1], [9]:

Theorem 4.5. *Given two germs of hypersurfaces V_f and V_g . Let $\mathbb{P}C(V_f)$, respectively $\mathbb{P}C(V_g)$, denote the projectivized tangent cone which is a subvariety of $\mathbb{C}P^{n-1}$. If $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f)) \neq 0$ and $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_g)) \neq 0$, then topological equisingularity of V_f and V_g implies $m_f = m_g$.*

The key point of the proof is that: if $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f)) \neq 0$ then by theorem (4.3), $m_f = \inf\{s \in \mathbb{N} | \Lambda(h^s) \neq 0\}$.

It is unknown whether $\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_f))$ is a topological invariant or not.

Example 4.6. Let $g = z_1^l + z_2^l + \dots + z_n^l$, $V_g = g^{-1}(0) \subset \mathbb{C}^n$ and $C(V_g) \subset \mathbb{C}P^{n-1}$ and F_θ be the fibre of the Milnor fibration of g at the origin. By (6.1) in the appendix we have

$$\mu_g = (l-1)^n.$$

With one blowing up at the origin, the singularity of g may be resolved and then we may apply the theorem (4.3): $S = \mathbb{C}P^{n-1}$ and

$$S_m = \begin{cases} \phi & \text{if } m \neq l \\ \mathbb{C}P^{n-1} \setminus V_g & \text{if } m = l. \end{cases}$$

By theorem (4.3),

$$\mu_g = (-1)^{n-1}[-1 + l\chi(S_l)].$$

The numbers $\chi(S_l)$ and $\chi(C(V_g))$ are related by

$$\chi(S_l) + \chi(C(V_g)) = \chi(\mathbb{C}P^{n-1}) = n.$$

Therefore we obtain the following well known formula

$$\chi(C(V_g)) = n - \frac{1 - (1-l)^n}{l}.$$

Example 4.7. Let \mathcal{A} be the set of all holomorphic functions g such that $0 \in \mathbb{C}^n$ is an isolated singularity for the first nonzero homogeneous part of the Taylor expansion of g . Then by an argument (see (6.3) in the appendix) the origin is an isolated singularity of g . Let $g \in \mathcal{A}$ with algebraic multiplicity l and the leading term g_l . Since g and g_l have the same projectivized tangent cones and V_{g_l} is topologically equivalent to $V_{z_1^l + z_2^l + \dots + z_n^l}$ then by the previous example

$$\chi(\mathbb{C}P^{n-1} \setminus \mathbb{P}C(V_g)) = \frac{1 - (1-l)^n}{l} \neq 0.$$

Hence by theorem (4.5) any topological equivalence between two elements of \mathcal{A} preserves multiplicities.

Still it is unknown whether there is any topological equivalence between $g \in \mathcal{A}$ and $f \notin \mathcal{A}$. By contrary if there exists such an equivalence then $k < l$, where k and l are the multiplicities of f and g respectively. The reason is that by (6.2) the Milnor number $\mu_f > (k-1)^n$ and $\mu_g = (l-1)^n$ and by theorem (4.3), Milnor number is a topological invariant. Therefore it remains to show that: Let $g = z_1^l + z_2^l + \dots + z_n^l$ and $f = f_k + \dots + f_{k+r}$ with $k < l$ and $k+r \geq l$ then the germs V_f and V_g at the origin are not topologically equisingular.

5. ON THE DEFORMATION OF COMPLEX HYPERSURFACES

Let us, instead of dealing with a pair of hypersurfaces, consider families of hypersurfaces, V_{f_t} , all having an isolated singular point at the origin and depending continuously in $t \in [0, 1]$ and $f_0 = f$ and $f_1 = g$. We denote by $C(V_{f_t})$, the tangent cone at 0 of V_{f_t} , that is, the zero set of the initial polynomial of f_t . H. King generalized theorem (2.4) as follows [11]:

Theorem 5.1. *Suppose $f_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, $t \in [0, 1]$ is a continuous family of holomorphic germs with the same Milnor number and $n \neq 3$. Then there is a continuous family of germs of homeomorphisms $h_t : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ so that $f_0 = f_t \circ h_t$*

Now we have the following result:

Theorem 5.2. *If for every $t_0 \in [0, 1]$ there exist a neighborhood I_{t_0} of t_0 in $[0, 1]$ and a line L_{t_0} through 0 in \mathbb{C}^n such that $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then topological equisingularity of the family implies equimultiplicity provided that $n \neq 3$.*

Proof. By theorem (5.1) there exists a continuous family of homeomorphisms φ_t such that $f_t = f \circ \varphi_t$. Therefore for every $t_0 \in [0, 1]$ we may write

$$f_s = f_{t_0} \circ \varphi_{t_0 s}$$

where $\varphi_{t_0 s} = \varphi_{t_0}^{-1} \circ \varphi_s$. Since f_{t_0} is uniformly continuous on a compact small ball $B_r \subset \mathbb{C}^n$ around 0, there exists $\eta > 0$ such that, for any $z, w \in B_r$,

$$|z - w| < \eta \implies |f_{t_0}(z) - f_{t_0}(w)| < \min_{u \in S_\delta} |f_{t_0}(u)|,$$

where S_δ is the boundary of \overline{D}_δ , the closed disc with radius $\delta < r/2$ in L_{t_0} around 0. Let $\varepsilon := \min\{\eta, \delta\}$. By continuity of φ_s , if I_{t_0} is sufficiently small then $|\varphi_{t_0 s}(z) - z| < \varepsilon$ for $s \in I_{t_0}$. Then for all z in the closed ball $B_\delta \subset \mathbb{C}^n$, $\varphi_{t_0 s}(z) \in B_r$ and

$$|f_{t_0}(z) - f_{t_0} \circ \varphi_{t_0 s}(z)| < \min_{u \in S_\delta} |f_{t_0}(u)|.$$

In particular for all $z \in S_\delta$ we have

$$|f_{t_0}(z) - f_s(z)| < |f_{t_0}(z)|, \text{ for } s \in I_{t_0}.$$

By hypothesis $L_{t_0} \cap C(V_{f_s}) = \{0\}$ for $s \in I_{t_0}$, then m_{f_s} is the order at 0 of $f_s|_{L_{t_0}}$ for $s \in I_{t_0}$. By theorem (1.6) in [12, Ch.VI], $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ have the same number of zeros, counted with their multiplicities in the interior of \overline{D}_δ . As $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ vanish only at 0 on \overline{D}_δ , the orders at 0 of $f_{t_0}|_{L_{t_0}}$ and $f_s|_{L_{t_0}}$ are equal. So $m_{f_{t_0}} = m_{f_s}$ for $s \in I_{t_0}$. This tells us that the multiplicity of the deformation is constant. \square

6. TOPOLOGICAL INVARIANTS FOR HOLOMORPHIC VECTOR FIELDS

Let $U \subset \mathbb{C}^n$ be an open neighborhood of $0 \in \mathbb{C}^n$ and $X : U \rightarrow \mathbb{C}^n$, $X(0) = 0$, a holomorphic vector field with a singularity at $0 \in \mathbb{C}^n$. The integral curves of X are complex curves i.e. Riemann surfaces parametrized locally as the solutions of the differential equation

$$\frac{dx}{dt} = X(x), \quad x \in U, \quad t \in \mathbb{C}.$$

These curves define a complex one dimensional foliation $\mathcal{F} = \mathcal{F}_X$ with singularity at $0 \in \mathbb{C}^n$. We define the *algebraic multiplicity* of X as the degree of its first nonzero jet, i.e. $m = m_X$ where

$$X = X_m + X_{m+1} + \cdots$$

is the Taylor series of X and X_m is not identically zero.

In analogy with the case of hypersurfaces we define the *Milnor number* of the vector field X at $0 \in \mathbb{C}^n$ as

$$\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle X_1, \dots, X_n \rangle}$$

where \mathcal{O}_n is the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$ and $\langle X_1, \dots, X_n \rangle$ is the ideal generated by the germs at $0 \in \mathbb{C}^n$ of the coordinate functions of X . This number is finite if and only if $0 \in \mathbb{C}^n$ is an isolated singularity of X , a hypothesis which we will assume from now on.

We say \mathcal{F}_X is topologically equivalent with $\mathcal{F}_{X'}$, X' is a holomorphic vector field defined in a neighborhood U' of $0 \in \mathbb{C}^n$, if there is a homeomorphism $\varphi : U \rightarrow U'$ fixing the origin (singularity) and sending every leaf of the foliation \mathcal{F}_X into a leaf of the foliation $\mathcal{F}_{X'}$.

A similar question is the following: is m_X a topological invariant of the foliation \mathcal{F}_X ?

In a remarkable work [5], C. Camacho, A. Lins Neto and P. Sad deal with this problem. First of all they prove the following result:

Theorem 6.1. *The Milnor number of a holomorphic vector field X as above is a topological invariant provided that $n \geq 2$.*

Now we restrict ourselves to $n = 2$ and recall some of the features coming from [5]. A germ of a vector field X with an isolated singularity may be given by $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ where a and b are holomorphic functions with isolated zero in a neighborhood $U \subset \mathbb{C}^2$. Denote by \mathcal{F} the foliation induced by X . The main tool in the local study is the resolution theorem of Seidenberg [21] that establishes a canonical reduction. More precisely, there is a holomorphic map $\pi : M \rightarrow \mathbb{C}^2$ obtained as a composition of a finite number of blowing ups at points over $\{0\}$, such that at each singular point $m \in M$ of $\tilde{\mathcal{F}}$ the foliation with isolated singularity constructed from $\pi^*(\omega)$ is reduced: there are coordinate charts (z, w) such that $z(m) = w(m) = 0$, $\tilde{\mathcal{F}}$ is given locally by the expression $A(z, w)\frac{\partial}{\partial x} + B(z, w)\frac{\partial}{\partial y} = 0$ and the Jacobian $\frac{\partial(A, B)}{\partial(z, w)}(0, 0)$ has at least one nonzero eigenvalue. Moreover if $\lambda_1 \neq 0 \neq \lambda_2$ are eigenvalues of the matrix then $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$. If one of the eigenvalues of the above matrix of a singularity is zero and another different from zero we call it saddle-node. By definition a *generalized curve* is a germ of a vector field X at $0 \in (\mathbb{C}^2, 0)$ and singular at the origin such that its desingularization does not admit any saddle-node. In [5] Camacho, Lins Neto and Sad proved that this property is invariant under topological equivalences and finally they deduced the following:

Theorem 6.2. *The algebraic multiplicity of a generalized curve is a topological invariant.*

Also we may say the same as theorem (3.2) for polynomial foliations.

7. APPENDIX

Let \mathcal{A} be the set of all holomorphic functions f such that $0 \in \mathbb{C}^n$ is an isolated singularity not only for f but also for the first nonzero homogeneous polynomial of the Taylor expansion of the f . Actually if the origin is an isolated singularity of the leading term of f then the same holds for f .

Remark 7.1. We have the following relation between multiplicity and Milnor number of f (see page 194 in [3]):

$$\mu_f = (m_f - 1)^n.$$

Remark 7.2. If $0 \in \mathbb{C}^n$ is not an isolated singularity of the first nonzero homogeneous polynomial then $\mu_f > (m_f - 1)^n$.

The following proposition is true in any dimension. But the following proof is based on the theorem of Lé and Ramanujam which is valid for $n \neq 3$.

Proposition 7.3. *The germ at the origin of the hypersurface defined by an element $f \in \mathcal{A}$ with the algebraic multiplicity k is topologically equivalent with the germ at the origin of the hypersurfaces defined by $z_1^k + \dots + z_n^k$.*

Proof. By a symbol $f \sim g$ between two germs of holomorphic functions at the origin we mean V_f and V_g , two germs of hypersurfaces defined by f and g respectively, are topological equivalent. Let

$$f = f_k + f_{k+1} + \dots$$

be the Taylor expansion of f , where f_i is homogeneous polynomial of degree i . The family $(H_t)_{t \in [0,1]} \in \mathcal{A}$:

$$H_t = f_k + t f_{k+1} + t^2 f_{k+2} + \dots$$

defines a μ -constant family between f and f_k . So $f \sim f_k$.

Now our task is to show $P(z) \sim (z_1^k + \dots + z_n^k)$ where $P(z)$ is a homogeneous polynomial of degree k .

Claim: *There is a non zero complex number α such that 0 is an isolated singularity of $F_t(z) := (1-t)(z_1^k + \dots + z_n^k) + t\alpha P(z)$ for $t \in [0, 1]$.*

The proof of claim: The partial derivatives of $F_t(z)$ form a system of bihomogeneous polynomials of bidegree $(1, k-1)$:

$$\begin{aligned} \frac{\partial F_t}{\partial z_1} &= k(1-t)z_1^{k-1} + t\alpha \frac{\partial P}{\partial z_1} \\ &\vdots \\ \frac{\partial F_t}{\partial z_n} &= k(1-t)z_n^{k-1} + t\alpha \frac{\partial P}{\partial z_n} \end{aligned}$$

and $V := \text{Zero}(\frac{\partial F_t}{\partial z_1}, \dots, \frac{\partial F_t}{\partial z_n})$ is an algebraic subset of $\mathbb{CP}(1) \times \mathbb{CP}(n-1)$.

Now consider the projection $\pi : \mathbb{CP}(1) \times \mathbb{CP}(n-1) \rightarrow \mathbb{CP}(1)$. Image of V , $\pi(V)$, is a Zariski-closed subset of $\mathbb{CP}(1)$ (see for instance [16] Pg. 33). Since $F_t(z)$ for $t = 0$ has the isolated singularity, $(1 : 0)$ is not in the $\pi(V)$. Therefore $\pi(V)$ is finite and there are infinitely many lines in the complement of $\pi(V)$ in $\mathbb{CP}(1)$. Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$ we may choose lines passing through the origin. This means that there is α such that the claim is true for every $t \in \mathbb{R}$.

□

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